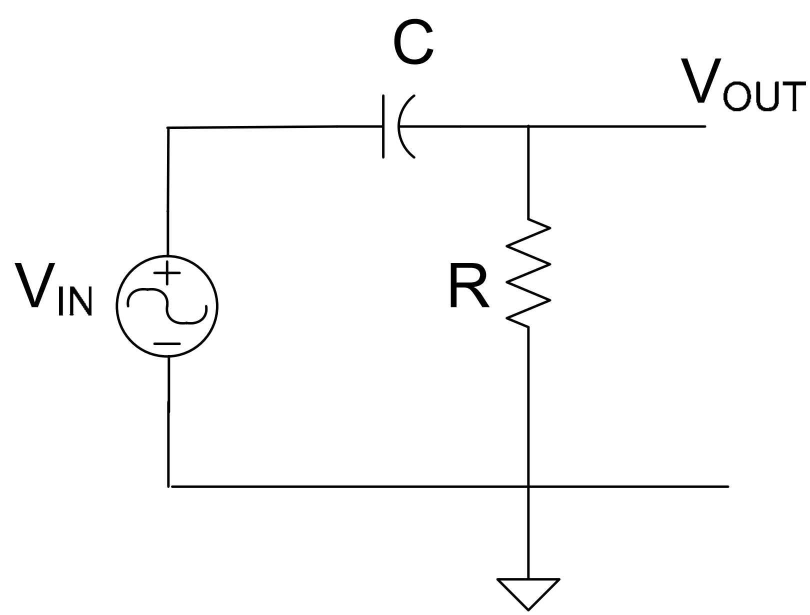
EE 230
Lecture 5

Linear Systems

- Poles/Zeros/Stability
- Stability

Quiz 4

Obtain the transfer function T(s) for the circuit shown. $\left(T(s) = \dfrac{V_{OUT}(s)}{V_{IN}(s)} \right)$



And the number is ?

1 3 8

5 4

2 6

9 7

And the number is ?

1 3 8

5 4

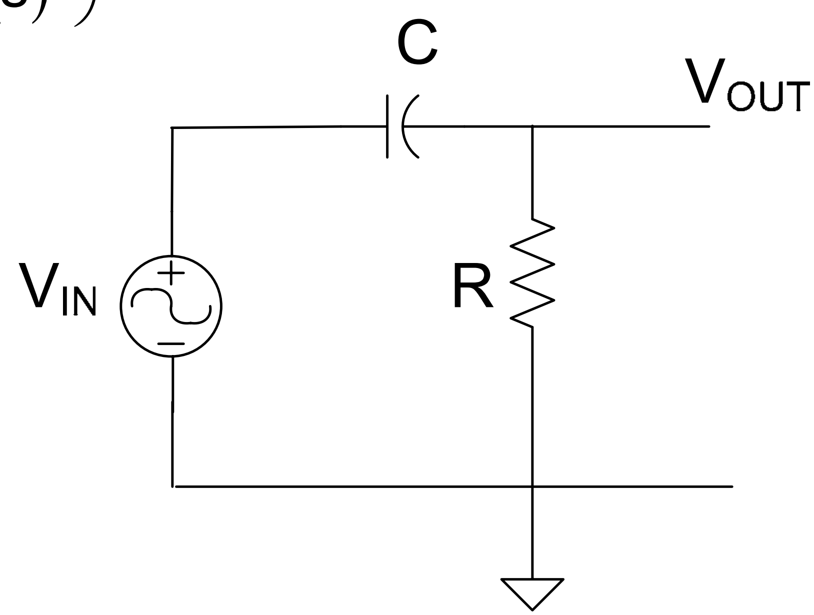
3

2 6

9 7

Quiz 4

Obtain the transfer function T(s) for the circuit shown. $\left(T(s) = \frac{V_{OUT}(s)}{V_{IN}(s)}\right)$



Solution:

$$T(s) = \frac{R}{R + \frac{1}{sC}} = \frac{RCs}{1 + RCs}$$

Review from Last Time

T(s) is often used instead of $T_P(j\omega)$ in the electronics and systems communities when characterizing the frequency response of linear circuits or systems

$$T(s)\Big|_{s=j\omega} = T_P(j\omega)$$

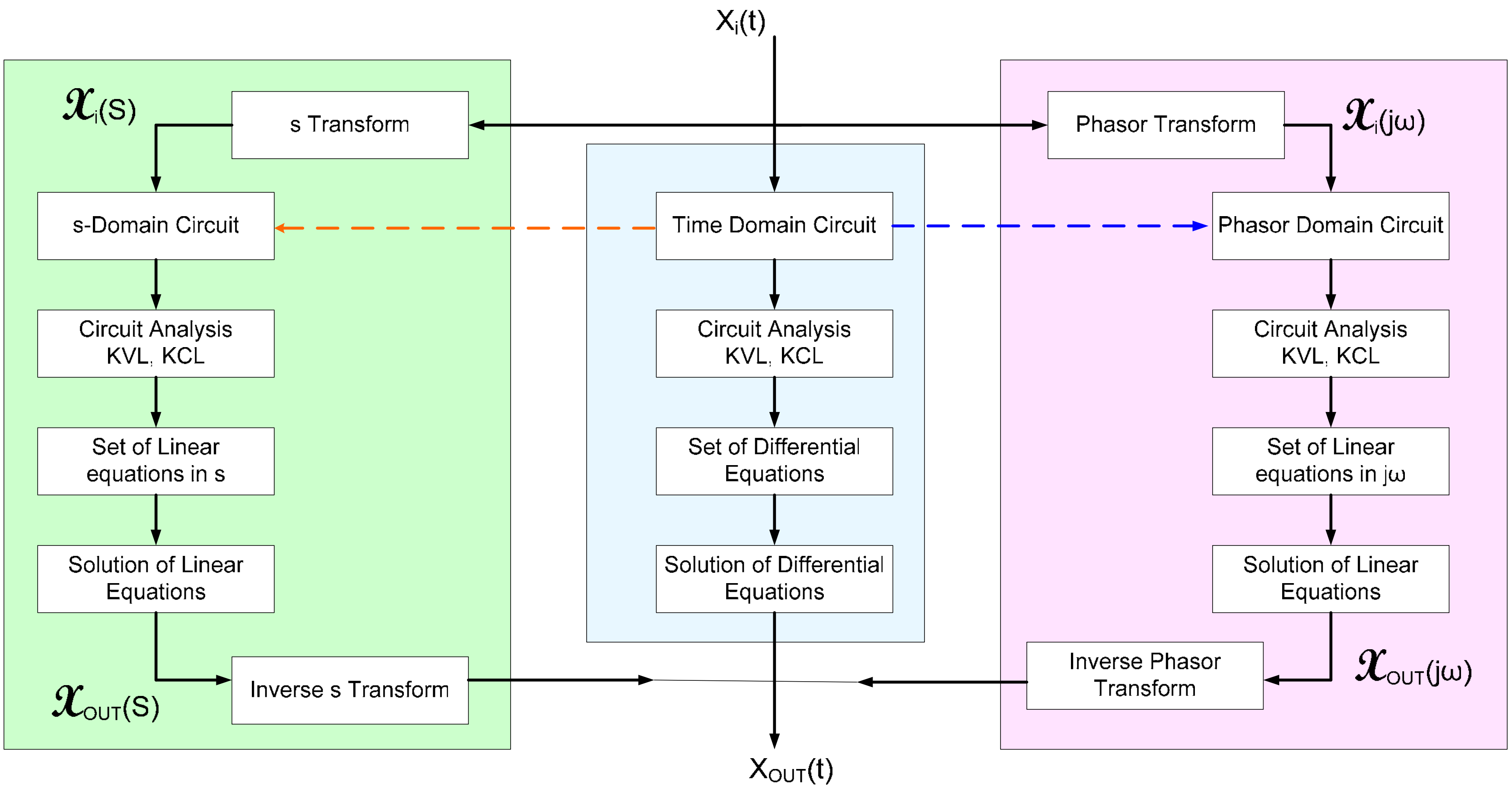
For linear system with excitation $V_m \sin(\omega t+\theta)$ and transfer function T(s)

$$V_{OSS} = V_M \left|T_P(j\omega)\right| \sin\left(\omega t + \theta + \gamma_{T_P(j\omega)}\right)$$

$$V_{OSS} = V_M \left|T(s)\Big|_{s=j\omega}\right| \sin\left(\omega t + \theta + \gamma_{T(s)|_{s=j\omega}}\right)$$

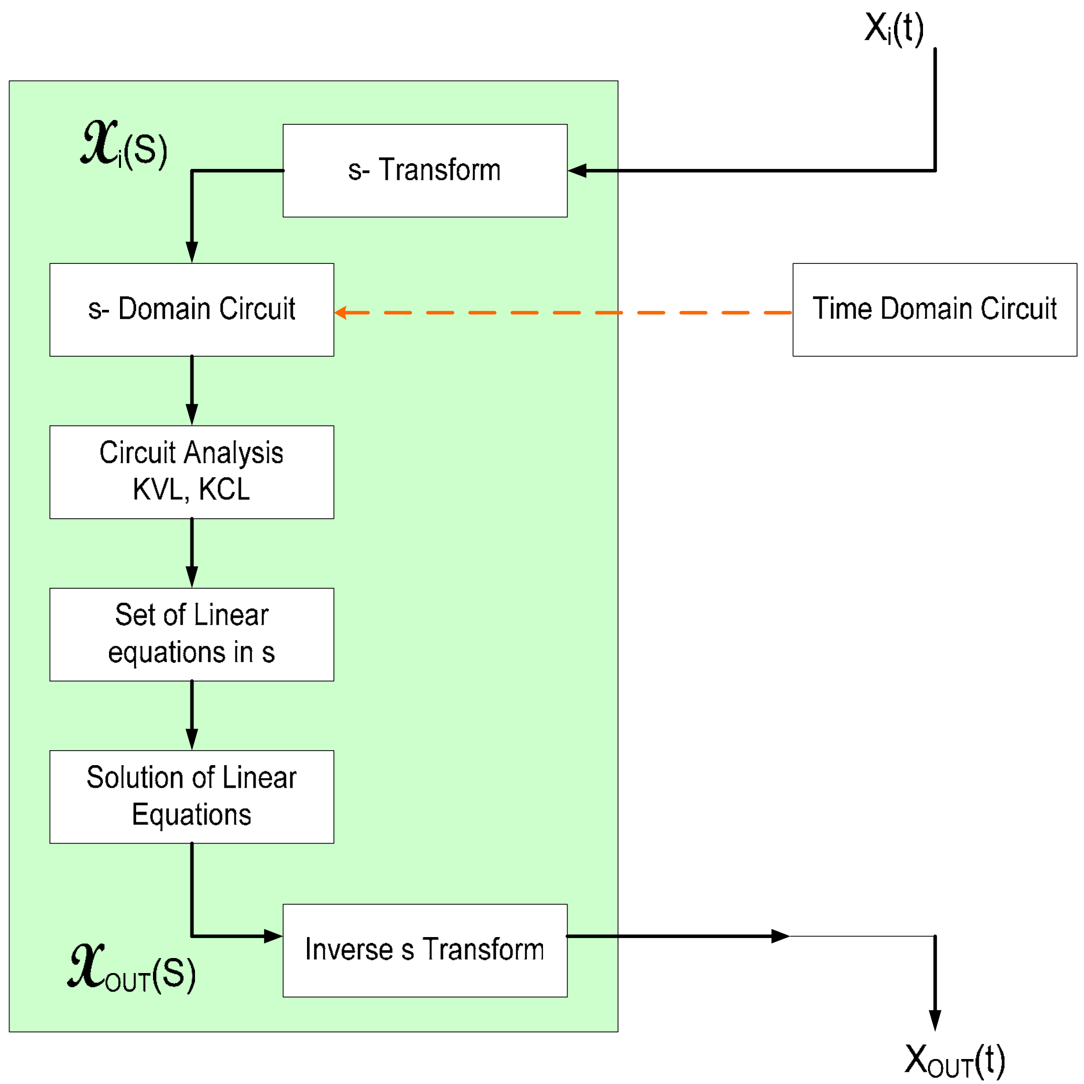
Review from Last Time

Time, Phasor, and s- Domain Analysis



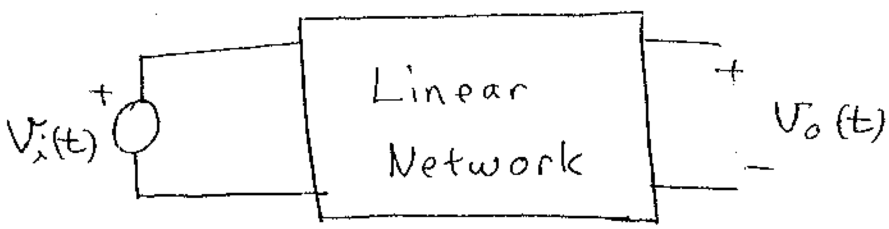
Review from Last Time

s- Domain Analysis



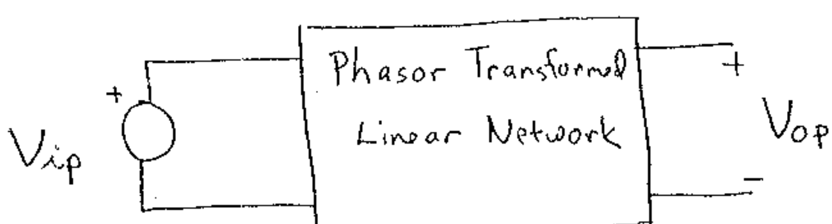
Sinusoidal Steady State Analysis of Linear Networks - A Review

Consider a linear network with input $V_i(t)$ and output $V_o(t)$



Assume $V_i(t) = V_m \sin(\omega t + \theta)$ (1)

Phasor Analysis



Phasor-transformed Circuit

Note: • V_{ip} and V_{op} are the phasor transforms of $V_i(t)$ and $V_o(t)$ respectively

• The phasor transformed linear network is obtained by making the following element transformations

— $C \rightarrow \frac{1}{j\omega C}$ (2)

— $L \rightarrow j\omega L$ (3)

— All other elements unchanged

Analyzing the phasor-domain network, we obtain the output phasor as the product of the input phasor and a complex function, $T_p(j\omega)$, determined by the network. This can be written as

$$V_{op} = T_p(j\omega) \cdot V_{ip} \tag{4}$$

$T_p(j\omega)$ can be written in polar form as

$$T_p(j\omega) = |T_p(j\omega)| \angle \arg(T(j\omega)) \tag{5}$$

where $\arg(T(j\omega))$ is the angle of the function $T_p(j\omega)$

The input phasor, from (1), can be written as

$$V_{ip} = V_m \angle \Theta \tag{6}$$

Substituting (5) and (6) into (4), we obtain the output phasor

$$V_{op} = (V_m \angle \Theta)(|T_p(j\omega)| \angle \arg(T(j\omega))) \tag{7}$$

which can be rewritten in standard polar form as

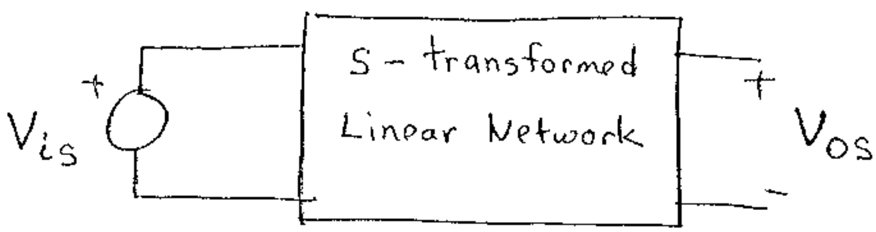
$$V_{op} = [V_m \cdot |T(j\omega)|] \angle (\Theta + \arg(T(j\omega))) \tag{8}$$

The sinusoidal steady state output can be obtained from the inverse phasor transform of V_{OP} in (8)

$$\boxed{V_o(t) = V_m \, |T_p(j\omega)| \sin(\omega t + \theta + \arg(T(j\omega)))} \quad (9)$$

Equation (9) is a key result!

S-domain analysis



S-transformed Circuit

Note
- V_{is} and V_{OS} are the s-transforms of $V_i(t)$ and $V_o(t)$ respectively
- The s-transformed linear network is obtained by making the following element transformations
  - $C \rightarrow \dfrac{1}{SC}$ (2')
  - $L \rightarrow SL$ (3')
  - All other elements unchanged

Analyzing the s-domain network, we obtain the s-domain output as the product of the s-domain input and an s-domain function, $T(s)$, determined by the network. This can be written as

$$V_{os} = T(s) \cdot V_{is} \qquad (4')$$

The sinusoidal steady state response can be obtained from the inverse-S transform of V_{os} which becomes, after transient response terms are neglected,

$$\boxed{V_o(t) = V_m \, |T(j\omega)| \sin(\omega t + \theta + \arg(T(j\omega)))} \qquad (9$$

where $T(j\omega) = T(s)\big|_{s=j\omega}$

Equation (9') is a key result!

Note: $\underline{T(s) \text{ is called the system transfer function}}$

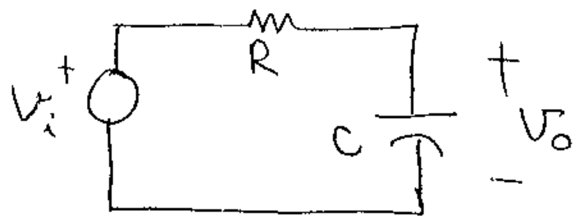
Note: $T_p(j\omega) = T(s)\big|_{s=j\omega}$

Note: $T_p(s) = T(j\omega)\big|_{\omega=\frac{s}{j}}$

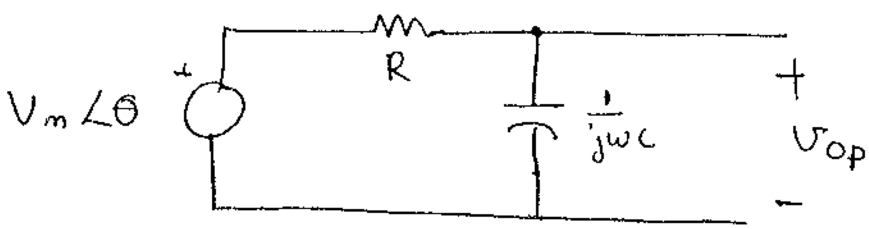
Note: $T_p(s) = T(s)$

Note: Equations (9) and (9') are identical

Example: Obtain the transfer function and the sinusoidal steady state response of the following circuit using a) phasor analysis ; b) s-domain analysis and c) differential equations. Assume $V_i = V_m \sin(\omega t + \theta)$



a) Phasor analysis



Phasor-transformed circuit

By voltage divider

$$V_{op} = \left(\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \right) V_m \angle \theta \qquad (1)$$

$$V_{op} = \left(\frac{1}{1 + RC\omega j} \right) V_m \angle \theta \qquad (2)$$

From (2), $T_p(j\omega) = \frac{V_{op}}{V_{ip}} = \frac{V_{op}}{V_m \angle \theta} = \frac{1}{1 + RC\omega j}$ (3)

$$\therefore \quad T(s) = \frac{1}{1 + RCs} \qquad (4)$$

Converting $T_p(j\omega)$ in (2) from rectangular to polar form, we obtain

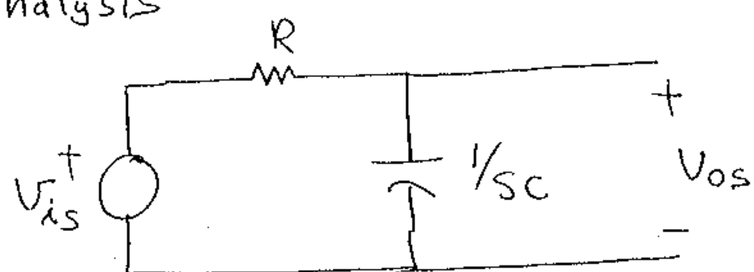
$$V_{op} = \left(\frac{1}{\sqrt{1+R^2C^2\omega^2}} \angle -\tan^{-1}\omega RC\right) V_m \angle\theta \qquad (5)$$

$$V_{op} = \frac{V_m}{\sqrt{1+R^2C^2\omega^2}} \angle(\theta - \tan^{-1}\omega RC) \qquad (6)$$

Taking the inverse phasor transform of (6) we obtain the sinusoidal steady state response

$$V_o(t) = \frac{V_m}{\sqrt{1+R^2C^2\omega^2}} \sin(\omega t + \theta - \tan^{-1}(\omega RC)) \qquad (7)$$

b) s-domain Analysis



s-transformed circuit

By voltage divider

$$V_{os} = \left[\frac{\frac{1}{sC}}{R + \frac{1}{sC}}\right] V_{is} \qquad (8)$$

$$V_{os} = \left[\frac{1}{1+RCs}\right] V_{is} \qquad (9)$$

It follows from (9) that the transfer function is

$$T(s) = \frac{1}{1+RCs} \qquad (10)$$

From (10),

$$T(j\omega) = \frac{1}{1+RC\omega j} \qquad (11)$$

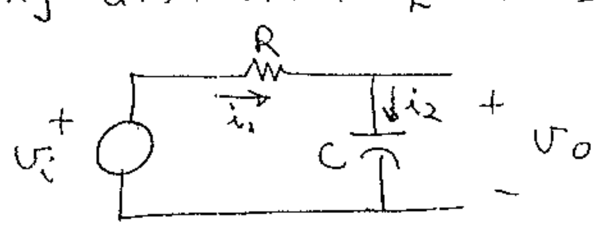
Thus the sinusoidal steady state output is given by

$$V_o(t) = V_m \, |T(j\omega)| \sin(\omega t + \theta + \angle T(j\omega)) \qquad (12)$$

from (12), this becomes

$$V_o(t) = \frac{V_m}{\sqrt{1+R^2C^2\omega^2}} \sin(\omega t + \theta - \tan^{-1}(\omega RC)) \qquad (13$$

c) using differential equations



time-domain circuit

$$i_1 = \frac{V_i - V_o}{R} \qquad (14)$$

$$i_2 = C\frac{dv_o}{dt} \qquad (15)$$

equating i_1 and i_2, we obtain the differential equation

$$\frac{V_i - V_o}{R} = C\frac{dv_o}{dt} \qquad (16)$$

I will use Laplace transforms to solve. Taking the Laplace transform of (16)

$$\frac{V_{is} - V_{os}}{R} = C\,s\,V_{os} \qquad (17)$$

Simplifying (17), we obtain

$$V_{os} = \frac{1}{1+RCS}\cdot V_{is} \qquad (18)$$

If $V_{is} = V_m \sin(\omega t + \theta)$, taking the inverse Laplace transform and neglecting the transient response, we obtain

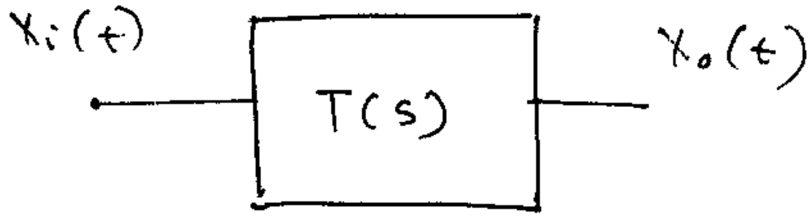
$$V_o(t) = V_m\,|T(j\omega)|\,\sin(\omega t + \theta - \arg(T(j\omega))) \qquad (1$$

where $T(j\omega) = \dfrac{1}{1+RC\omega j}$ (20)

Thus,

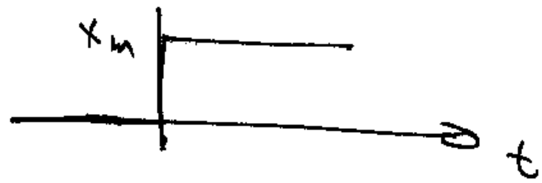
$$V_o(t) = \frac{V_m}{\sqrt{1+R^2C^2\omega^2}}\sin(\omega t + \theta - \tan^{-1}\omega RC) \qquad (21)$$

Step Response of First-Order Networks



$$T(s) = \frac{N(s)}{s+p}$$

If $x_i = X_m u(t)$

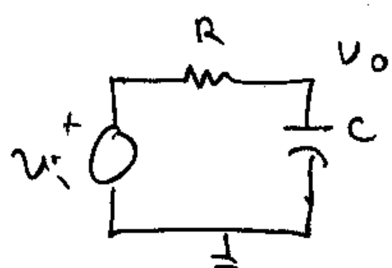


$$x_o(t) = F + (I - F)\, e^{-t/\tau}$$

$$\tau = p^{-1}$$

Example:

$v_i = V_m u(t)$



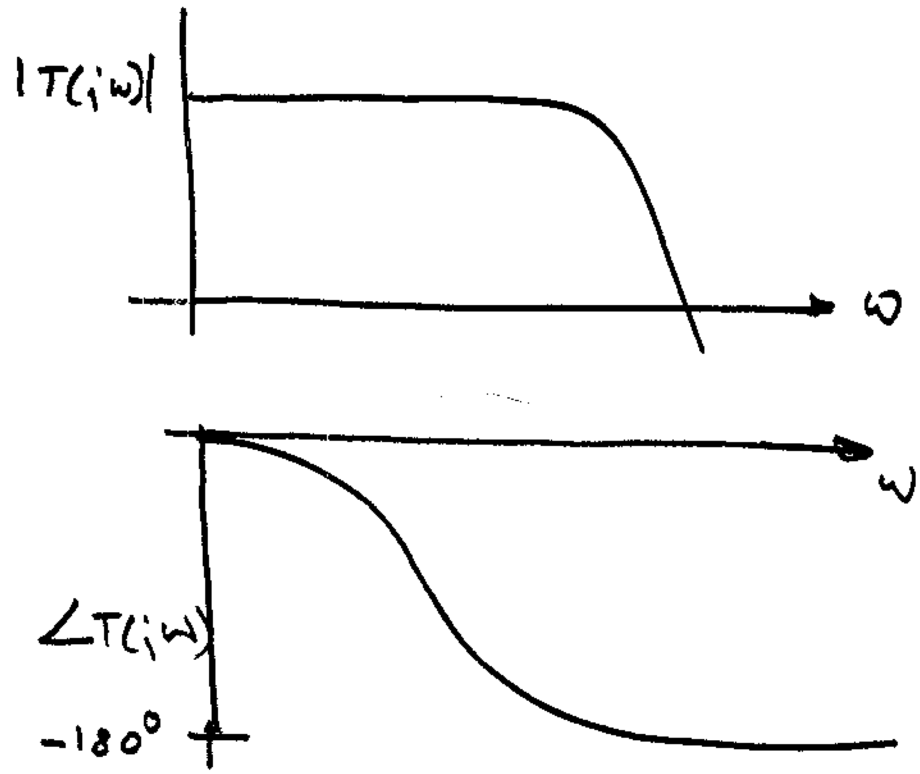
$$\frac{v_o}{v_i} = \frac{1}{1+RCs} \rightarrow p = \frac{1}{RC}$$

$$v_o(t) = V_m + -V_m e^{-\frac{t}{RC}} = V_m\left(1 - e^{-t/RC}\right)$$

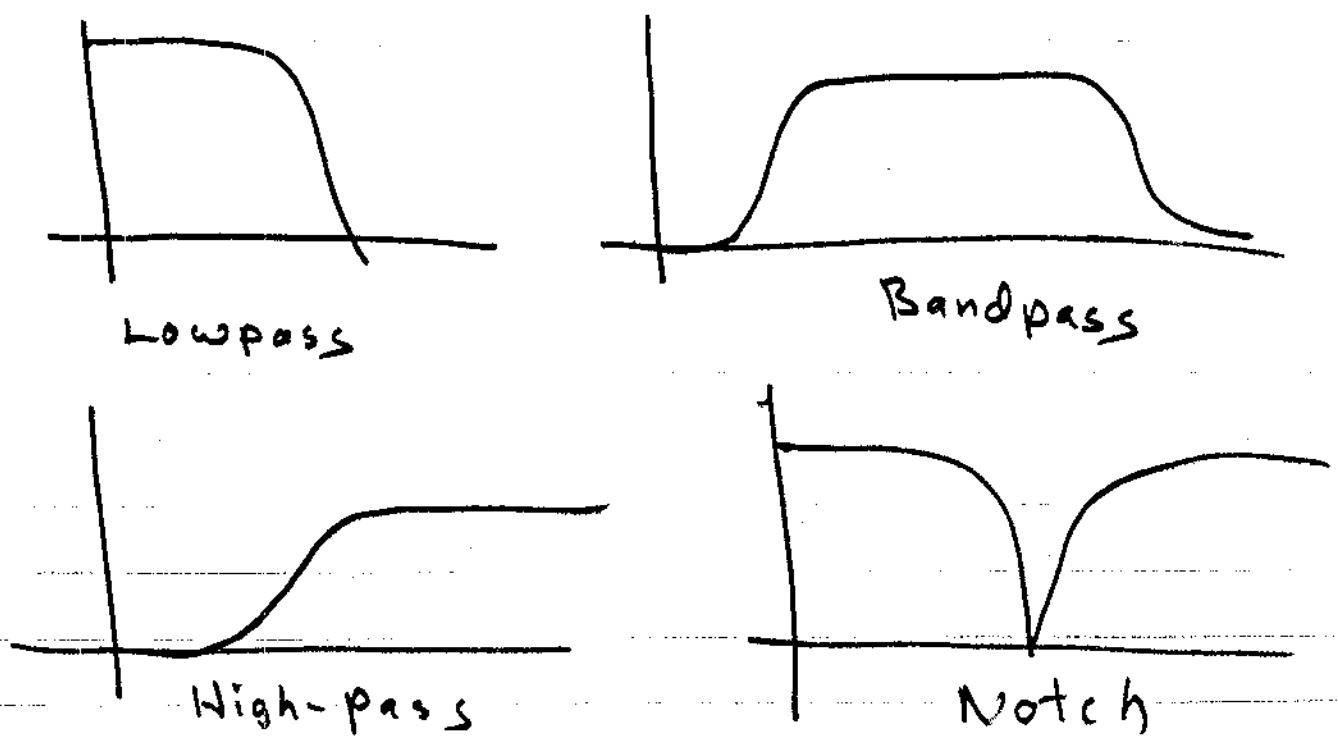
"Gain" of linear networks

- $T(j\omega)$ represents the gain of a linear network with a sinusoidal input
- $T(j\omega)$ is frequency dependent
- $|T(j\omega)|$ is termed the magnitude of the gain
- $\angle T(j\omega)$ is termed the phase or angle of the gain
- Plots of $|T(j\omega)|$ and $\angle(T(j\omega))$ often used to characterize the network
- These plots characterize the "frequency response" of the network

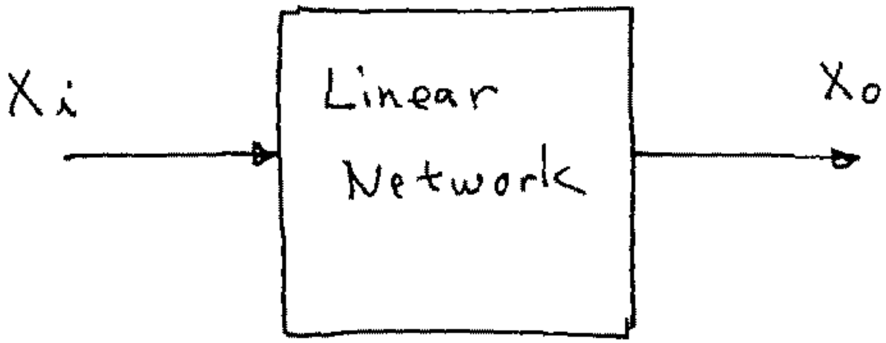
Example:



Nomenclature



Recall:



$$T(s) = \frac{\vec{X}_o(s)}{\vec{X}_i(s)}$$

Claim: $T(s)$ can always be expressed as a rational fraction in s if the network has a finite number of lumped components

will only be interested in networks with finite number of lumped components in this course

$$T(s) = \frac{N(s)}{D(s)}$$

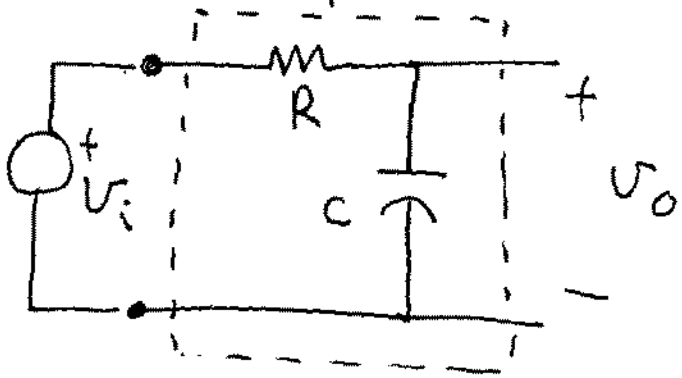
where $$N(s) = \sum_{k=0}^{m} a_k s^k$$

$$D(s) = \sum_{k=0}^{n} b_k s^k$$

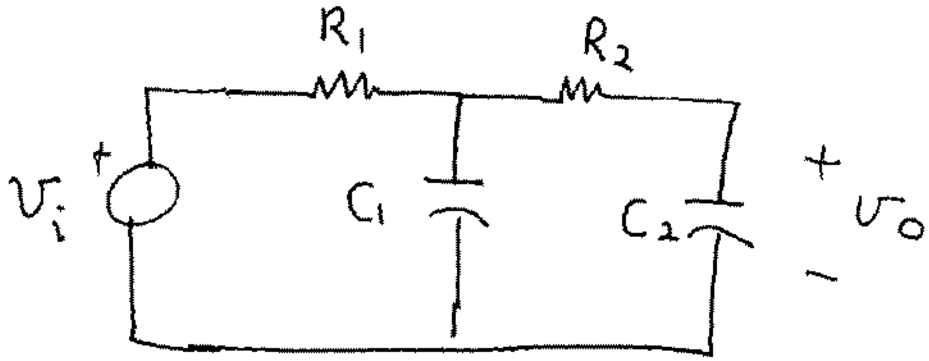
The roots of $D(s)$ are the poles of the network

The roots of $N(s)$ are the zeros of the network

Example:



$$\frac{V_o}{V_i} = \frac{1}{1+RCS}$$



$$\frac{v_o}{v_i} = \frac{\frac{1}{R_1R_2C_1C_2}}{S^2 + S\left[\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{C_2R_2}\right] + \frac{1}{R_1R_2C_1C_2}}$$

Example: $$T(s) = \frac{s+4}{s^2+9s+8}$$

$$= \frac{s+4}{(s+1)(s+8)}$$

Zeros: $\{ s = -4 \}$

Poles: $\{ s = -1, \; s = -8 \}$

Often designated as

$$z = -4$$

$$p_1 = -1, \quad p_2 = -8$$

Claim: A system with a 1st order lowpass transfer function with a pole p and dc gain K has a unit step response of

$$r(t) = F + (I - F)e^{pt}$$

I: Initial Value
F: Final Value

$\therefore$ p^{-1} is the time constant of the system

Alternate representations of Transfer Functions

$$T(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=1}^{m} a_i s^i}{\sum_{i=1}^{n} b_i s^i}$$

$$T(s) = \frac{A_1 \prod_{i=1}^{m} (s - z_i)}{B_1 \prod_{i=1}^{n} (s - P_i)}$$

If $X(s)$ is the input

$$X_o(s) = X(s)\,\frac{A_1}{B_1}\,\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

$$X_o(s) = \frac{\Theta_1}{s-p_1} + \frac{\Theta_2}{s-p_2} + \cdots \frac{\Theta_n}{s-p_n} + \frac{\beta_1}{(s-\hat{p}_1)} + \cdots \frac{\beta_k}{(s-\hat{p}_k)}$$

$$\mathcal{L}^{-1}(X_o(s)) = \Theta_1 e^{p_1 t} + \Theta_2 e^{p_2 t} + \cdots \Theta_n e^{p_n t} + \beta_1 e^{\hat{p}_1 t} + \cdots \beta_k e^{\hat{p}_k t}$$

Theorem: Any network comprised of Rs, Ls, Cs, will have all poles in the L.H.P.

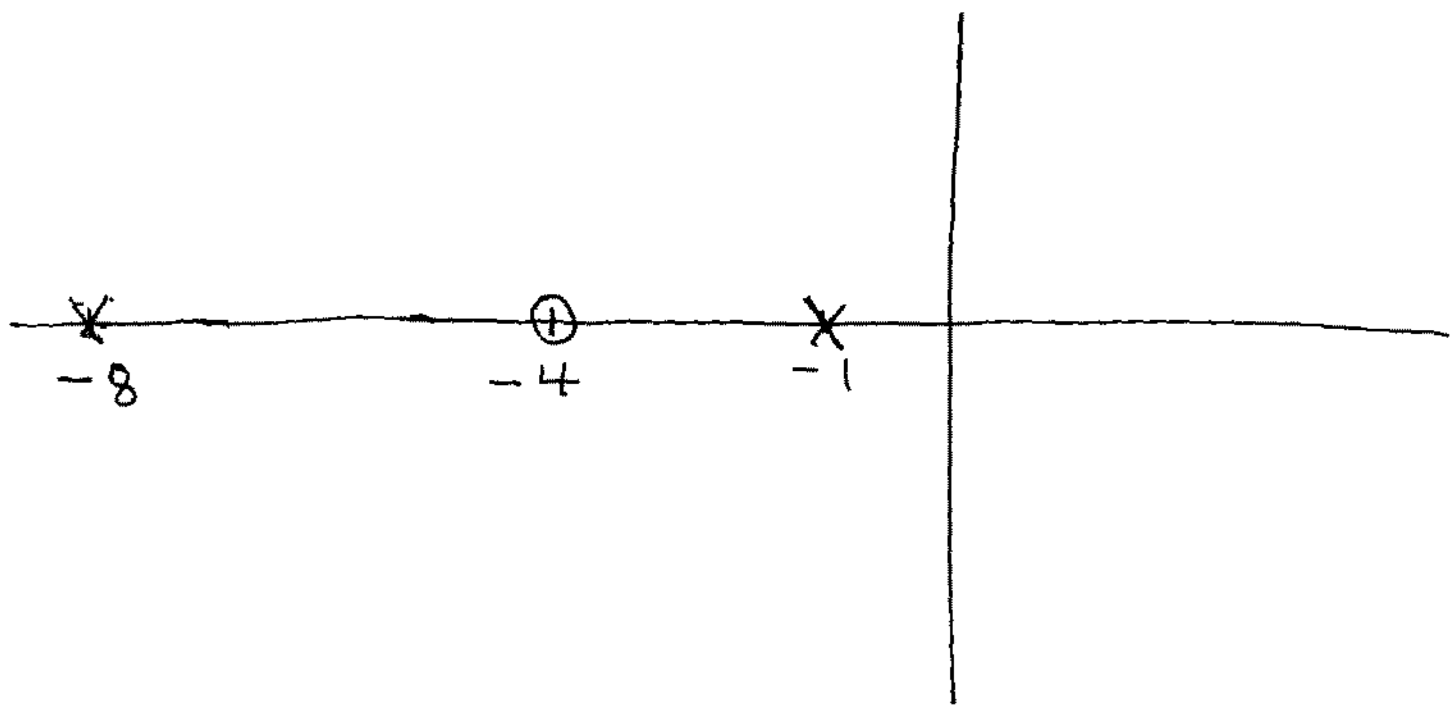
Note: This theorem is not true, in general, if amplifiers (dependent sources) are included.

End of Lecture 5

Pole - Zero Plots

- Plots of the poles and zeros in the complex plane ($X \rightsquigarrow$ poles, $O \rightsquigarrow$ zeros)

Example: $z = -4$, $P_1 = -1$, $P_2 = -8$



Example $T(s) = \dfrac{s-1}{s^2+2s+2}$

Determine poles and zeros and plot in complex plane

$z_1 = 1$

$p = ?$

$$s^2+2s+2=0 \Rightarrow s = \frac{-2 \pm \sqrt{2^2-(4)(2)}}{2}$$

$$s = \frac{-2 \pm \sqrt{-4}}{2}$$

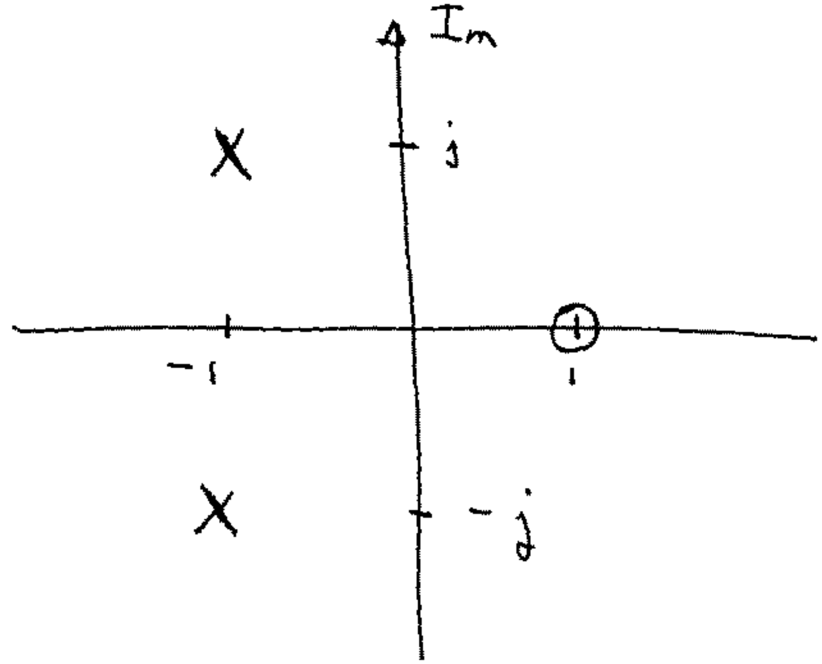
$$s = -1 \pm j$$

$$S = -1 \pm j$$

$$P_1 = -1 + j$$

$$P_2 = -1 - j$$

$$Z_1 = 1$$



Example: $$T(s) = \frac{4s+1}{s^5 + s^4 + 2s^3 + 3s^2 + 2s + 1}$$

Zeros: $\left\{-\frac{1}{4}\right\}$

poles: – 5 in number

– closed form expression does not exist for polynomials of order 5 or higher

Theorem: A system is stable iff
<u>all</u> poles lie in the left half-plane

What is "stable"?

- A linear system is stable iff any bounded input will result in a bounded output

- A system is stable iff the output due to any appropriately small input does not cause the output to go to $\pm\infty$ and does not create an output that persists indefinitely

Examples:

$$T(s) = \frac{1}{s+1}$$

pole at $p = -1$

$\therefore$ system is stable

step response : $r(t) = F + (I - F)e^{-t/\tau} = e^{-t}$

$$T(s) = \frac{1}{s-1}$$

pole at $p = 1$

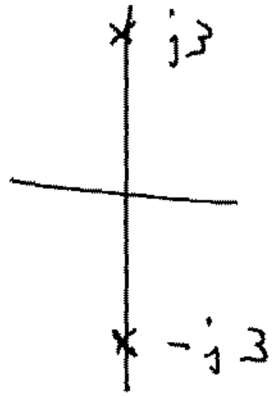
$\therefore$ system is unstable

It can be shown that the step response is

$$r(t) = e^{t}$$

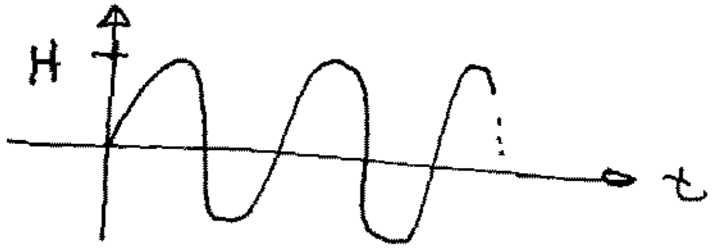
$r(t)$ diverges to ∞ !

Example: $T(s) = \dfrac{4}{s^2+9}$ Poles: $\{-j3, +j3\}$



It can be shown that the step response will include a term

$$r(t) = H \sin 3t$$



$\therefore$ Output persists indefinitely

Is stability good or bad?

- depends on what is desired

Is instability good or bad?

- depends upon what is desired